

$$\nabla \vec{E} = -4\pi \left(\vec{\nabla} \rho + \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right)$$

$$\nabla \vec{B} = \frac{4\pi}{c} \vec{\nabla} \times \vec{J}$$

Hence,

$$\vec{E}(\vec{x}, t) = - \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \left[\vec{\nabla}' \rho + \frac{1}{c} \frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}}$$

$$\vec{B}(\vec{x}, t) = \frac{1}{c} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} [\vec{\nabla}' \times \vec{J}]_{\text{ret}}$$

Note:

$$[f(\vec{x}', t')]_{\text{ret}} = f(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c})$$

In particular,

$$\vec{\nabla}' [f]_{\text{ret}} \neq [\vec{\nabla}' f]_{\text{ret}}$$

\uparrow \uparrow
 fixed t fix t'

(and \vec{x} is fixed).

Use the chain rule!

Consider $w = f(x', t')$, where $t' = g(x', t)$

$$\left(\frac{\partial w}{\partial x'}\right)_t = \left(\frac{\partial w}{\partial x'}\right)_{t'} + \left(\frac{\partial w}{\partial t'}\right)_{x'} \left(\frac{\partial t'}{\partial x'}\right)_t$$

$$\left(\frac{\partial w}{\partial t}\right)_{x'} = \left(\frac{\partial w}{\partial t'}\right)_{x'} \left(\frac{\partial t'}{\partial t}\right)_{x'}$$

Here $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$

So,

$$(\vec{\nabla}' g)_t = (\vec{\nabla}' g)_{t'} + \left(\frac{\partial g}{\partial t'}\right)_{x'} (\vec{\nabla}' t')_t$$

i.e., $\vec{\nabla}'[g]_{\text{ret}} = [\vec{\nabla}' g]_{\text{ret}} + \left[\frac{\partial g}{\partial t'}\right]_{\text{ret}} \vec{\nabla}' \left(-\frac{|\vec{x} - \vec{x}'|}{c}\right)$

Define $\vec{R} = \vec{x} - \vec{x}'$

$$R = |\vec{R}| = |\vec{x} - \vec{x}'|$$

$$\vec{\nabla}'_{x'} = -\vec{\nabla}_R = -\hat{R} \frac{\partial}{\partial R} \quad \text{in spherical coordinates}$$

$$\vec{\nabla}' \left(-\frac{|\vec{x} - \vec{x}'|}{c}\right) = \frac{\vec{R}}{c}$$

$$\boxed{[\vec{\nabla}' g]_{\text{ret}} = \vec{\nabla}'[g]_{\text{ret}} - \frac{\vec{R}}{c} \left[\frac{\partial g}{\partial t'} \right]_{\text{ret}}}$$

$$[\vec{D}' \times \vec{J}]_{\text{ret}} = \vec{D}' \times [\vec{J}]_{\text{ret}} + \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \times \hat{\vec{R}} \frac{c}{c}$$

↑ order of cross
product reversed
(extra sign)

$$\vec{E}(\vec{x}, t) = - \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \left\{ \vec{D}' [\rho]_{\text{ret}} + \frac{1}{c^2} \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} - \hat{\vec{R}} \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \right\}$$

using

$$\vec{D}' \frac{1}{|\vec{x} - \vec{x}'|} = - \vec{D} \frac{1}{|\vec{x} - \vec{x}'|} = - \hat{\vec{R}} \frac{\partial}{\partial R} \left(\frac{1}{R} \right) = \hat{\vec{R}} \frac{1}{R^2}$$

where $\hat{\vec{R}} = \vec{x} - \vec{x}'$

Integration by parts yields:

$$\vec{E}(\vec{x}, t) = \int d^3x' \left\{ \frac{\hat{\vec{R}}}{R^2} [\rho(\vec{x}', t')]_{\text{ret}} + \frac{\hat{\vec{R}}}{cR} \left[\frac{\partial \rho(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} \right\}$$

Likewise,

$$\vec{B}(\vec{x}, t) = \frac{1}{c} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \left\{ \vec{D}' \times [\vec{J}]_{\text{ret}} + \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \times \hat{\vec{R}} \frac{1}{c} \right\}$$

Integrate by parts to obtain

$$\boxed{\vec{B}(\vec{x}, t) = \frac{1}{c} \int d\vec{x}' \left\{ [\vec{J}(\vec{x}', t')]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\}}$$

Telfimenco equations

These are the time-dependent generalization
of the Coulomb and Biot-Savart laws.

By the second chain rule,

$$\frac{\partial}{\partial t} [\rho]_{\text{ret}} = \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} \left(\frac{\partial t'}{\partial t} \right)_{\vec{x}'} = \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}}$$

$\stackrel{1}{=} 1$ since

Recall that

$$t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

$$\vec{D} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\vec{D}' \cdot [\vec{J}]_{\text{ret}} = [\vec{D}' \cdot \vec{J}]_{\text{ret}} + \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \cdot \vec{D}' \left(-\frac{R}{c} \right)$$

$$= - \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} + \frac{1}{c} \left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \cdot \hat{R}$$

$$\int d^3x' \frac{\hat{R}}{cR} \left[\frac{\partial \rho}{\partial t'} \right]_{\text{ret}} = -\frac{1}{c} \int d^3x' \frac{\vec{D}' \cdot [\vec{J}]_{\text{ret}} \hat{R}}{R}$$

$$+ \frac{1}{c^2} \int d^3x' \frac{\left(\left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \cdot \hat{R} \right) \hat{R}}{R}$$

Integrating by parts ($\hat{R} = \vec{R}/R$)

$$\frac{\vec{D}' \cdot [\vec{J}]_{\text{ret}} \hat{R}}{R} = \vec{D}' \cdot \left(\frac{[\vec{J}]_{\text{ret}} \vec{R}}{R^2} \right) + \frac{[\vec{J}]_{\text{ret}} - 2([\vec{J}]_{\text{ret}} \cdot \hat{R}) \hat{R}}{R^2}$$

Second term above is equivalent to

$$\frac{-([\vec{J}]_{\text{ret}} \cdot \hat{R}) \hat{R} + ([\vec{J}]_{\text{ret}} \times \hat{R}) \times \hat{R}}{R^2}$$

End result:

(Papofsky and Phillips)

$$\boxed{\vec{E}(\vec{x}, t) = \int d^3x' \left\{ \frac{\hat{R}}{R^2} [g(\vec{x}', t')]_{\text{ret}} \right.}$$

$$\left. + \frac{1}{cR^2} \{ ([\vec{J}]_{\text{ret}} \cdot \hat{R}) \hat{R} + ([\vec{J}]_{\text{ret}} \times \hat{R}) \times \hat{R} \} \right.$$

$$\left. + \frac{1}{c^2 R} \left(\left[\frac{\partial \vec{J}}{\partial t'} \right]_{\text{ret}} \times \hat{R} \right) \times \hat{R} \right\}}$$

Radiation fields

$$\vec{E}_{\text{rad}}(\vec{x}, t) = \frac{1}{c^2} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \left(\left[\frac{\partial \vec{J}(\vec{x}; t')}{\partial t'} \right]_{\text{ret}} \times \hat{\vec{R}} \right) \times \hat{\vec{R}}$$

$$\vec{B}_{\text{rad}}(\vec{x}, t) = \frac{1}{c^2} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}} \times \hat{\vec{R}}$$

At large $|\vec{x}|$, $\hat{\vec{R}} \approx \hat{n}$ where $\hat{n} = \frac{\vec{x}}{|\vec{x}|}$

$$\Rightarrow \boxed{\vec{E}_{\text{rad}} = \vec{B}_{\text{rad}} \times \hat{n} \quad \text{for large } |\vec{x}|}$$

Radiation of EM waves

Let d be a typical dimension of the sources (currents and charges). The wavelength of the waves is $\lambda = \frac{2\pi c}{\omega}$. Typically, $d \ll \lambda$.

example: $d \sim 1 \text{ \AA}$ (atomic dimension)

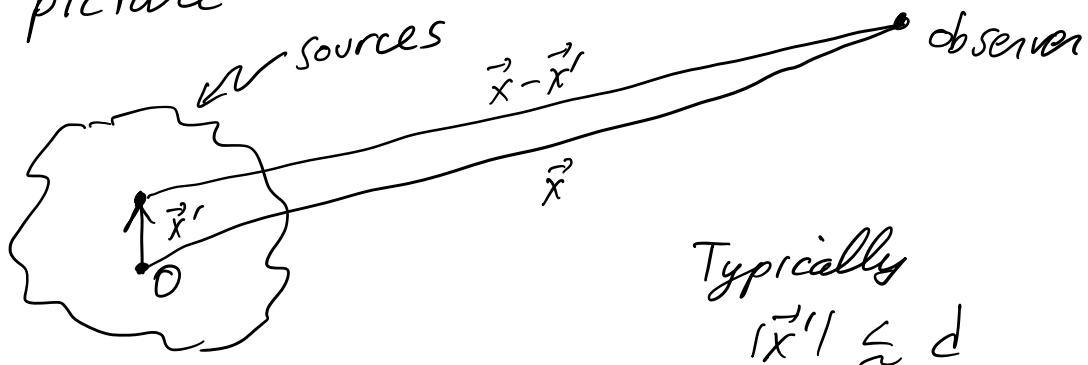
$\lambda \sim 5000 \text{ \AA}$ (visible light)

Three "zones"

$$r = |\vec{x}|$$

1. Near (static) zone $d \ll r \ll \lambda$
2. Intermediate (induction) zone $d \ll r \sim \lambda$
3. Far (radiation) zone $d \gg \lambda \gg r$

The picture



$$\begin{aligned} r &= |\vec{x}| \\ r' &= |\vec{x}'| \end{aligned} \quad \vec{R} = \vec{x} - \vec{x}', \quad R = |\vec{R}|.$$

Harmonic sources

Complex notation

$$\vec{J}(\vec{x}', t') = \vec{J}(\vec{x}') e^{-i\omega t'}$$

$$\vec{B}_{\text{rad}} = -\frac{1}{c^2 r} \hat{n} \times \int d^3 x' \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \right]_{\text{ret}}$$

using $\hat{R} \approx \hat{n}$. for large r .

$$= \frac{i\omega}{c^2 r} \hat{n} \times \int d^3 x' \vec{J}(x') e^{ik|\vec{x}-\vec{x}'|} e^{-i\omega t}$$

after using $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$

$$|\vec{x}-\vec{x}'| = [r^2 + r'^2 - 2\vec{x} \cdot \vec{x}']^{1/2}$$

$$= r \left[1 + \frac{r'^2 - 2\vec{x} \cdot \vec{x}'}{r^2} \right]^{1/2}$$

$$= r \left[1 - \frac{\hat{n} \cdot \vec{x}'}{r} + O\left(\frac{1}{r^2}\right) \right] \quad \hat{n} = \frac{\vec{x}}{r}$$

$$= r - \hat{n} \cdot \vec{x}' + O\left(\frac{1}{r}\right)$$

$$\vec{B}_{\text{rad}} = \frac{i\omega}{c^2 r} e^{i(kr - \omega t)} \hat{n} \times \int d^3 x' \vec{J}(x') e^{-ik\vec{x}' \cdot \hat{n}}$$

$$+ O\left(\frac{1}{r^2}\right)$$

$$\vec{E}_{\text{rad}} = \vec{B}_{\text{rad}} \times \hat{n}$$

$$e^{-ik\vec{x}' \cdot \hat{n}} \approx 1 - ik\vec{x}' \cdot \hat{n} + \dots \quad \text{since } kd \ll 1$$

The full expansion is called the multipole expansion.

Electric dipole approximation (E1) : $e^{-ik\vec{x}' \cdot \hat{n}} = 1$.

$$\vec{B}_{E1}(\vec{x}, t) = \frac{i\omega}{c^2 r} e^{i(kr - \omega t)} \hat{n} \times \int d^3 \vec{x}' \vec{J}(\vec{x}')$$

$$J_i' = \partial'_k (J_k x_i') - x_i' \vec{\nabla}' \cdot \vec{J} \quad (\text{Cartesian tensors})$$

$$\vec{\nabla}' \cdot \vec{J} + \frac{\partial \vec{S}}{\partial t} = 0 \quad \vec{J}(\vec{x}, t) = \vec{J}(\vec{x}) e^{-i\omega t}$$

$$\vec{\nabla}' \cdot \vec{J}(\vec{x}) = i\omega S(\vec{x}) \quad S(\vec{x}, t) = S(\vec{x}') e^{-i\omega t}$$

$$J_i(\vec{x}') = \partial'_k (J_k x_i') - x_i' S(\vec{x}') i\omega$$

$$\vec{p} = \int d^3 x' \vec{x}' S(\vec{x}') \quad \text{definition of electric dipole vector}$$

$$\vec{p}(t) = \vec{p} e^{-i\omega t}$$

$$\int d^3 x' \vec{J}(\vec{x}') = -i\omega \vec{p}$$

$$\boxed{\vec{B}_{E1}(\vec{x}, t) = \frac{k^2}{r} e^{i(kr - \omega t)} \hat{n} \times \vec{p}} \quad (k = \frac{\omega}{c})$$

$$\vec{E}_{E1}(\vec{x}, t) = \vec{B}_{E1} \times \hat{n} \quad (\hat{n} \times \vec{p}) \times \hat{n}$$

$$\boxed{\vec{E}_{E1}(\vec{x}, t) = \frac{k^2}{r} e^{i(kr - \omega t)} \vec{P}_\perp} \quad \begin{aligned} &= \vec{p} - \hat{n}(\hat{n} \cdot \vec{p}) \\ &= \vec{p} - \vec{P}_{||} = \vec{P}_\perp \end{aligned}$$

I can write the above as

$$\vec{E}_{E1}(\vec{r}, t) = \frac{k^2}{r} \vec{P}_1 \left(t - \frac{r}{c} \right)$$

The polarization of the E1 radiation is determined by \vec{E}_{E1} .

Energy transport as $r \rightarrow \infty$

$$P = \oint \vec{S} \cdot \hat{n} da \quad da = r^2 d\Omega$$

Power radiated into a solid angle $d\Omega$ averaged over a cycle

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \operatorname{Re} (\vec{E} \times \vec{B}^*)$$

Hence,

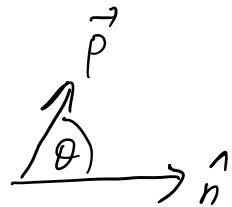
$$\begin{aligned} \frac{dP}{d\Omega} &= \langle \vec{S} \rangle \cdot \hat{n} r^2 \\ &= \frac{c}{8\pi} \operatorname{Re} [(\vec{B} \times \hat{n}) \times \vec{B}^*] \cdot \hat{n} r^2 \\ &= \frac{c}{8\pi} \operatorname{Re} [\vec{B} \cdot \vec{B}^* - (\hat{n} \cdot \vec{B})(\hat{n} \cdot \vec{B}^*)] r^2 \\ &= \frac{cr^2}{8\pi} |\hat{n} \times \vec{B}|^2 \quad |\vec{v}|^2 = \vec{v} \cdot \vec{v}^* \end{aligned}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^4}{8\pi} |\hat{n} \times (\hat{n} \times \vec{p})|^2}$$

$$= \frac{ck^4}{8\pi} \left[(\hat{n} \times \vec{p}) \cdot (\hat{n} \times \vec{p}^\perp) - \hat{n} \cdot (\hat{n} \times \vec{p}) \hat{n} \cdot (\hat{n} \times \vec{p}^\perp) \right]$$

↑
O

$$= \frac{ck^4}{8\pi} |\hat{n} \times \vec{p}|^2$$



$$= \frac{ck^4}{8\pi} |\vec{p}|^2 \sin^2 \theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^4}{8\pi} |\vec{p}|^2 \sin^2 \theta}$$

Total power radiated

$$P = \frac{ck^4}{8\pi} |\vec{p}|^2 2\pi \int_{-1}^1 d\cos \theta (1 - \cos^2 \theta)$$

$$\boxed{P = \frac{ck^4}{3} |\vec{p}|^2}$$